A Formal Theory of Triads Based on Non-Cooperative Games

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Abstract

We will investigate three person's social relations discussed by G.Simmel using game theory. Firstly, cooperative games are defined through non-cooperative games. Non-cooperative games treated in this paper are symmetric games appeared in the discussion of social dilemma situation or order problem in social sciences. We will introduce the notion of *strictly totally profitable games* which guarantee better individual benefit by cooperating with all members than individual benefit under non-cooperative situation. This condition is a sufficient condition for non-emptiness of the core of the coalitional game. As main results, we will prove that under some conditions there exist games which are strictly totally profitable or have empty core [1].

keywords: sociology of small groups, Simmel's proposition, triads, game theory,

1 Introduction

The study of three person groups in social sciences will be back to G.Simmel's works([24], 1907). Mills started his article([13], 1953) wrighting "In drawing his fundamental distinction between two-person groups and all groups of larger size, Simmel called attention to certain characteristics of the three-person situation." Also it is Mills who first pointed out that there is a closed relation between Simmel's small group theory and von Neumann-Morgenstern's game theory. He wrote "In this respect, Simmel and von Neumann and Morgenstern share common ground. One assumes and the others plan

[1]The essential part of this paper has been announced at the 41-st Conference of the Japanese Association for Mathematical Sociology (2006.3.3-4.) held at University of Tokyo.

for an elementary differentiating tendency in the three-some: namely, segregation into a *pair* and an *other*"(p.351r). In fact, von Neumann-Morgenstern's famous book([31]), they argued "The following seems worth noting: coalitions occur first in a zero-sum game when the number of participants in the game reaches three. In a two-person game there are not enough players to go around: a coalition absorbs at least two players, and then nobody is left to oppose. But while the three-person game of itself implies coalitions, the scarcity of players is still such as to circumscribe these coalitions in a definite way: a coalition must consist of precisely two players and be directed against precisely one (the remaining) player, "(p.221, footnote 2) or "It seems worth emphasizing that this characteristically "social" phenomena occurs only in the case of three or more participants."(p.225, footnote 2.) They also seems to be very much concerning about social problems since they are wrighting at Preface to First Edition that "The applications are of two kinds: On the one hand to games in the proper sense, on the other hand to economic and sociological problems which, as we hope to show, are best approached from this direction.,,, Our major interest is of course, in the economic and sociological direction."

Unlike Milles who investigated empirical data in the laboratory, Caplow([2]) theoretically analyzed the coalition pattern of the triad. He examined the models of the triad whose members are not identical in power. He classified logically the three person's relations $A \geq B \geq C$, and $A > (B + C)$ or $A < (B + C)$ into 6 types, but unlike our setting, he never considered game theoretic relationships between each member.

Unfortunately, not only von Neumann-Morgenstern but also Mills himself did not developed their arguments more precisely on the fields of sociological phenomena by using game theory[2].

It is Nakano([19],[20]) who first tried to formulate concretely the relation between Simmel's triads and von Neumann-Morgenstern's coalitional game theory[3] . He primarily asserted that Simmel's proposition "Tendency to divide into a coalition of two members against the third" correspondes to emptiness of the core of three person cooperative games and von Neumann-Morgenstern's stability solutions, especially discriminatory solutions correspond to Simmel's three different types of situations,i.e. "the non-partisan and the mediator", "the tertius gaudens" and "divide et impera". But he also noticed

[2]We have a criticism from theoretical point of view that von Neumann-Morgenstern always firstly discussed zero-sum games then, generalized them to non-zero sum n person games by reducing zero-sum $(n + 1)$ person game adding a damy player, but this formulation , though theoretically simple, is not suitable for applying social phenomena because the damy player does not have any meaning in social context(cf. [31],p221 footnote. "the genreral n-person game is closely related to the zero-sum $n+1$-person game."). As Suzuki([28]) pointed out, the word "core" does not appear in von Neumann-Morgenstern's book supposedly becouse any zero-sum game have empty core, but in this report, we will show that even if the begining non-cooperative game is in social dilemma situation, the cooperative game has possibly the core (see our Theoem 1 and also the footnote of the next page).

[3]von Neumann-Morgenstern([31]) wrote in the preliminary survey of zero-sum four-person games (Chapter VII), "Indeed, it will be noted that the interpretaion of the mathematical results of this phase leads quite naturally to specific "social" concepts and formulations."

the possibility that even if the core is not empty, three person group may split into two and one through bargaining. His assersion is a little bit ambiguous and it seems to be not so crealy characterized the situations.

The major objective of our investigation is to formulate three person groups based on no-cooperative game theory. Non-cooperative games investigated in this paper are so-called social dilemma games which are described as involving a conflict between "individual rationality" and "group rationality" which appeared in the context of Hamburger([8]), Schelling([25]), Daws([5],[6]), Muto([16],[17]) and so many peaple.

2 Cooperative games defined from non-cooperative games

First of all, we will formulate a cooperative game with a transferable utility or a coalitional game with side payment by defining a charastaristic function from a non-cooperative game. Our formulation is essentially equivalent to those of [21],[30],[22] but we rewright it by using random variables defined on an abstract probability space $(\Omega,\, \mathcal{F},\, P)$ for the reason that so-called mixed strategies are nathing but random variables.

Let $N = \{1,\, 2, \ldots,\, n\}$ be a set constituting n players. To avoid triviality, we assume the number of players is greater than or equal to 3 (i.e. $|N| \geq 3$). Let $S_i\,;\, i \in N$ and $u_i(\mathbf{s})\,;\, i \in N,\, \mathbf{s} = (s_1, \ldots, s_n) \in \Pi_{i \in N} S_i$ be a player i's strategy set[4] and a utility function respectively. For a subset T of N, $-T$ means a complementary set $N - T$ and we denote by $s_T = \{s_i\,;\, i \in T\}$ a strategy set of players' set T. Then we can represent a strategy set $\mathbf{s}$ of all players by $\mathbf{s} = (s_T,\, s_{-T})$ and the utility function $u_i(\mathbf{s})$ by $u_i(s_T,\, s_{-T})$.

Now for any non-empty subset T, we can set up a random variable X_T taking its values on a set $\Pi_{i \in T} S_i$ defined on an abstract probability space $(\Omega,\, \mathcal{F},\, P)$. We denote by $\mathcal{L}_T$ a set of the all possible T-valued random variables. According to von Neumann-Morgenstern, a coaliton means a subset T of N and a profit $v(T)$ for this coalition (so-called a characteristic function) is defined through the max-min principle as follows [5]:

[4]Here we assume that each S_i is a finite set but this assumption is not essential. To define random variables (=mixed strategies) , it is enough to assume that S_i is a measurable space.

[5]Harsanyi([9], p.214) criticizes an adaptation of a maximin strategy to define a characteristic function saying "But this model is open to the following objection: Why should either side expect the other side to choose the strategy of the highest damaging power irrespective of the costs to itself? This may be a very natural expectation in a constant-sum game, where one side's loss is necessarily the other side's gain, and vice versa, so that each side must cause the highest possible damage to the other side in order to maximize its own joint payoff. But in a variable-sum game one would rather expect that each side would try to find a suitable compromise between trying to maximize the costs of a conflict to the other side and trying to minimize the costs of a conflict to itself - in other words, between trying to minimize the joint payoff of the opposing coalition and trying maximize the joint payoff of their own coalition." We agree his comment but at least mathematical point of view, only a maximin strategy guarantees superadditivity of the characteristic function.

Definition 1[6]. We set $v(\emptyset) = 0$ and for $N \supset T \neq \emptyset$ and $\neq N$,

$$v(T) = \sup_{X_T \in \mathcal{L}_T} \inf_{X_{-T} \in \mathcal{L}_{-T}} \sum_{i \in T} E[u_i(X_T,\, X_{-T})],$$

and

$$v(N) = \sup_{X_N \in \mathcal{L}_N} \sum_{i \in N} E[u_i(X_N)],$$

where we assume that two random variables X_T and X_{-T} are independent.

This game is so-called a cooperative game with side payment. In this formulation, as von Neumann-Morgenstern themselves wrote([31], p.240), "$v(T)$ describes what a given coalition of players (specifically, the set T) can obtain from their opponents(the set $-T$)..., but it fails to describe how the proceeds of the enterprise are to be divided among the partners k belonging to T." Also they wrote(p.231) "If two players, say 1 and 2, decide to cooperate completely..., postponing temporarily, for a later settlement, the question of distribution, i.e. of the compensations to be paid between partners...." We also do not or can not discuss how to distribute their profit among the members in the coalition T, but in this paper we suppose that they will distribute their profit equally among them.

Proposition 1. (superadditivity of $v(T)$, [21] p.190, [22] Lemma in p.169, [30] Theorem 7.1.)

$$v(T \cup U) \geq v(T) + v(U)$$

holds for any subset T and U such that $T \cap U = \emptyset$.

Proof. First we note that

$$\begin{aligned} v(T \cup U) &= \sup_{X_{T\cup U} \in \mathcal{L}_{T\cup U}} \inf_{X_{-(T\cup U)} \in \mathcal{L}_{-(T\cup U)}} \sum_{i \in T\cup U} E[u_i(X_{T\cup U},\, X_{-(T\cup U)})] \\ &\geq \sup_{X_T \in \mathcal{L}_T} \sup_{X_U \in \mathcal{L}_U} \inf_{X_{-(T\cup U)} \in \mathcal{L}_{-(T\cup U)}} (\sum_{i \in T} E[u_i(X_T,\, X_U,\, X_{-(T\cup U)})] \\ &\quad + \sum_{i \in U} E[u_i(X_T,\, X_U,\, X_{-(T\cup U)})]), \end{aligned}$$

here we can assume that two random variables X_T and X_U are independent and also both are independent of $X_{-(T\cup U)}$ since $(X_T,\, X_U)$ is in $\mathcal{L}_{T\cup U}$ which is independent of $\mathcal{L}_{-(T\cup U)}$.

Since $(X_U,\, X_{-(T\cup U)}) \in \mathcal{L}_{-T}$ and is independent of X_T, and analogously $(X_T,\, X_{-(T\cup U)}) \in \mathcal{L}_{-U}$ and is independent of X_U, by definition of "inf", we have

$$\inf_{X_{-(T\cup U)} \in \mathcal{L}_{-(T\cup U)}} (\sum_{i \in T} E[u_i(X_T,\, X_U,\, X_{-(T\cup U)})] + \sum_{i \in U} E[u_i(X_T,\, X_U,\, X_{-(T\cup U)})])$$

[6] This definition is essentially the same as that of ([21],p.189). ([22],p.169), ([30].p.167). Suzuki([30]) and Harsanyi([9]) call this type of characteristic function a von Neumann-Morgenstern characteristic function though, in the original book([31]), they are discussing very vague way in case of zero-sum n person games. Ruce-Liffa([12], Chapter 8) axiomatically defined characteristic funcions and they do not discuss cooperative games defined from non-cooperative games.

$$\geq \inf_{X_{-T}\in\mathcal{L}_{-T}} \sum_{i\in T} E[u_i(X_T,\, X_{-T})] + \inf_{X_{-U}\in\mathcal{L}_{-U}} \sum_{i\in U} E[u_i(X_U,\, X_{-U})].$$

Since the above inequality holds for any $X_T \in \mathcal{L}_T$ and $X_U \in \mathcal{L}_U$, and X_T is independent of $(X_U,\, X_{-(T\cup U)}) = X_{-T} \in \mathcal{L}_{-T}$ and X_U is independent of $(X_T,\, X_{-(T\cup U)}) = X_{-U} \in \mathcal{L}_{-U}$, we finally get

$$\begin{aligned} v(T\cup U) &\geq \sup_{X_T\in\mathcal{L}_T} \inf_{X_{-T}\in\mathcal{L}_{-T}} \sum_{i\in T} E[u_i(X_T,\, X_{-T})] + \sup_{X_U\in\mathcal{L}_U} \inf_{X_{-U}\in\mathcal{L}_{-U}} \sum_{i\in U} E[u_i(X_U,\, X_{-U})] \\ &= v(T) + v(U). \qquad\qquad (Q.E.D) \end{aligned}$$

We remark that by superadditivity, for any non-empty subset T,

$$v(T) \geq \sum_{i\in T} v(\{i\})$$

holds, which means that each player may have some motivation to co-operate with some other members as far as they accept max-min principle, but for having explicit motivation, we need some more condition which is a little bit stronger condition than non-emptiness of a set of the core.

Definition 2. Totally Profitable Game.[7] A coalitional game $(N,\, v)$ with $v(N) \geq 0$ is called *totally profitable* if and only if

$$\frac{v(N)}{|N|} \geq \frac{v(T)}{|T|}$$

holds for any non-empty coalition. where $|T|$ denotes the total number of the elements of T.

Definition 3. Strictly Totally Profitable Game

A coalitional game $(N,\, v)$ with $v(N) \geq 0$ is called a *strictly totally profitable game* if and only if

$$\frac{v(N)}{|N|} > \frac{v(T)}{|T|}$$

holds for any non-empty coalition $T \neq N$[8],

By definition, if a game $(N,\, v)$ is totally profitable, then it is necessarily a balanced game which is known in the usual game theory. Therefore, any totally profitable game has non-empty core([25] , see also appendix). Obviously, in a (strictly) totally profitable game, each player, he or she has (strictly) better benefit under cooperation with all members than indivisual or partially coalitional profit.

[7]Suzuki-Nakamura([29]) called this a game having *"weak mean coalition power"* but our major concern is when strict inequality holds.

[8]If the game is not essential, then $v(N) = \sum_{i=1}^{n} v(\{i\})$ holds. Therefore, if a game is strictly totally profitable, then necessarily the game is essential.

3 Three person cooperative games defined by social dilemma games

Daws([5]) first gave a formal definition of social dilemma games considering that each of n players has a choice between two strategies D (for "Detecting") and C (for "Cooperating"). Let $D(m)$ be the player's payoff for a D choice when m players choose C, and let $C(m)$ be the payoff for a C choice when m players choose C, where the m refer to the number of players who choose C, not to a particular set of m players. He called a social dilemma game if $C(n) > D(0)$ and $D(m) > C(m+1)$ for $0 \leq m \leq n-1$ hold.

Muto([16], [17]) has intensively inspected the above games. He defines "cooperative situations" from the viewpoint of free ride and classified them.

Here after we assume that $|N| = n = 3$, that is, we will focus on three person games. In this case, we need 6 parameters $D(m)\,;\, m = 0, 1, 2$ and $C(m)\,;\, m = 1, 2, 3$. But first, for convenience, we assume

Assumption 1. $D(0) = 0$.

To describe the Nash equilibrium of our non-cooperate games, it is convenient to introduce the following parameters.
$\alpha_0 = D(0) - C(1), \quad \alpha_1 = D(1) - C(2), \quad \alpha_2 = D(2) - C(3)$
$\beta_1 = D(1) - C(1), \quad \beta_2 = D(2) - C(2)$

Equivalently, under Assumption 1, we have

$$C(1) = -\alpha_0, \quad C(2) = \beta_1 - \alpha_1 - \alpha_0, \quad C(3) = \beta_2 + \beta_1 - \alpha_2 - \alpha_1 - \alpha_0$$
$$D(0) = 0, \quad D(1) = \beta_1 - \alpha_0, \quad D(2) = \beta_2 + \beta_1 - \alpha_1 - \alpha_0$$

As usual, Nash equlibrium means a social situation where no one has a motivation to change his or her choice. Then a social situation in which m persons choose C is a Nash equilibrium if and only if $\alpha_{m-1} < 0$ and $\alpha_m > 0$ when $m = 1, 2$. A social situation in which no person chooses C is a Nash equilibrium if and only if $\alpha_0 > 0$ and a social situation in which every person chooses C is a Nash equilibrium if and only if $\alpha_2 < 0$.

Following Muto([17]), we give a table of 8 possible cases corresponding to the signature of $\alpha_i\,;\, i = 1, 2, 3$ and Nash equilibria. The classification is due to Muto when $n = 3$.

α_0	α_1	α_2	Nash equilibrium	classification
$+$	$+$	$+$	$m = 0$	Prisoner's dilemma
$+$	$+$	$-$	$m = 0,\, 3$	Assurance game
$+$	$-$	$+$	$m = 0,\, 2$	
$+$	$-$	$-$	$m = 0,\, 3$	Assurance game
$-$	$+$	$+$	$m = 1$	Chicken game
$-$	$+$	$-$	$m = 1,\, 3$	
$-$	$-$	$+$	$m = 2$	Chicken game
$-$	$-$	$-$	$m = 3$	No-conflict game

Now we focus on Muto's cooperative situation, that is,

Assumption 2. (Muto([17])
(i) Cost for cooperation: $D(m) > C(m)\,;\, m = 1,\, 2\ (\Longleftrightarrow \beta_1 > 0,\, \beta_2 > 0)$,
(ii)Guarantee for common profit: $D(0) < \max_{1 \le m \le 3} C(m)$.

To evaluate the value of our characteristic function v under our Assumptions, set

$$f(m) = mC(m) + (3 - m)D(m),\ 0 \le m \le 3.$$

Then, for $N = \{1,\, 2,\, 3\}$, we have

$$v(N) = \max_{0 \le m \le 3} f(m) (\ge f(0) = 0).$$

The relations between $f(m)$ and our parameters are the followings:
$f(0) = 3D(0) = 0$,
$f(1) = C(1) + 2D(1) = 2\beta_1 - 3\alpha_0$,
$f(2) = 2C(2) + D(2) = \beta_2 + 3\beta_1 - 3\alpha_1 - 3\alpha_0$,
$f(3) = 3C(3) = 3\beta_2 + 3\beta_1 - 3\alpha_2 - 3\alpha_1 - 3\alpha_0$.
When $T = \{i\}\,;\, i = 1,\, 2,\, 3$, we have

$$\begin{aligned} v(\{i\}) &= \sup_{\mathbf{X}_T} \inf_{\mathbf{X}_{-T}} E[u_i(\mathbf{X}_T,\, \mathbf{X}_{-T})] \\ &= \max\{\min_{1 \le m \le 3} C(m),\, \min_{0 \le m \le 2} D(m)\}. \end{aligned}$$

Now we have to evaluate $v(T)$ for $T = \{1,\, 2\}$. When X_T is fixed, the infimum attains at the end of the segments, therefore it is enough to check at $P(X_{-T} = C) = 1$ or $P(X_{-T} = D) = 1$, that is,

$$v(T) = \sup_{X_T} \min_{s_3} \sum_{i=1}^{2} E[u_i(X_T,\, s_3)].$$

Here we fix X_T, and set

$$I_C = \sum_{i=1}^{2} E[u_i(X_T,\, C)]$$

and

$$I_D = \sum_{i=1}^{2} E[u_i(X_T,\, D)].$$

More precisely, we have

$$\begin{aligned}
I_C &= \sum_{i=1}^{2} E[u_i(X_T,\, C)] \\
&= (\sum_{i=1}^{2} u_i(C,\, C,\, C))P(X_1 = C,\, X_2 = C) \\
&\quad + (\sum_{i=1}^{2} u_i(C,\, D,\, C))P(X_1 = C,\, X_2 = D) \\
&\quad + (\sum_{i=1}^{2} u_i(D,\, C,\, C))P(X_1 = D,\, X_2 = C) \\
&\quad + (\sum_{i=1}^{2} u_i(D,\, D,\, C))P(X_1 = D,\, X_2 = D).
\end{aligned}$$

By virtue of our formulations, we have

$$\begin{aligned}
I_C &= 2C(3)P(X_1 = C,\, X_2 = C) \\
&\quad + (C(2) + D(2))P(X_1 = C,\, X_2 = D \text{ or } X_1 = D,\, X_2 = C) \\
&\quad + 2D(1)P(X_1 = D,\, X_2 = D).
\end{aligned}$$

Analogously,

$$\begin{aligned}
I_D &= 2C(2)P(X_1 = C,\, X_2 = C) \\
&\quad + (C(1) + D(1))P(X_1 = C,\, X_2 = D \text{ or } X_1 = D,\, X_2 = C) \\
&\quad + 2D(0)P(X_1 = D,\, X_2 = D).
\end{aligned}$$

In order to evaluate the order between I_C and I_D, we consider the difference of I_C and I_D.

$$\begin{aligned}
I_C - I_D &= (2C(3) - 2C(2))P(X_1 = C,\, X_2 = C) \\
&\quad + (C(2) - C(1) + D(2) - D(1))P(X_1 = C,\, X_2 = D \text{ or } X_1 = D,\, X_2 = C) \\
&\quad + 2D(1)P(X_1 = D,\, X_2 = D) \\
&= 2(\beta_2 - \alpha_2)P(X_1 = C,\, X_2 = C) \\
&\quad + (\beta_2 + \beta_1 - 2\alpha_1)P(X_1 = C,\, X_2 = D \text{ or } X_1 = D,\, X_2 = C) \\
&\quad + 2(\beta_1 - \alpha_0)P(X_1 = D,\, X_2 = D).
\end{aligned}$$

Under our Assumptions, we classify our situation as a classification of the game.

Class I.

I-1. $\beta_1 - \alpha_0 \geq 0$ ($\Longleftrightarrow D(1) \geq 0$),

I-2. $\beta_2 + \beta_1 - 2\alpha_1 \geq 0$ ($\Longleftrightarrow C(2) + D(2) \geq C(1) + D(1)$),

I-3. $\beta_2 - \alpha_2 > 0$ ($\Longleftrightarrow C(3) > C(2)$).

In thie class, always $I_C \geq I_D$ holds regardless of the distribution of X_T. Therefore, in this class, we can easily determine the value of the characteristic function $v(\{1, 2\})$ and in consequence, we have the following theorem.

Theorem 1. Under Assumptions 1 and 2, if a game belongs to Class 1, then it is strictly totally profitable.

Proof. In this case, the characteristic function for $T = \{1, 2\}$ is, by definition,

$$v(T) = \sup_{X_T} I_D,$$

where

$$\begin{aligned} I_D &= 2C(2)P(X_1 = C, X_2 = C) \\ &\quad + (C(1) + D(1))P(X_1 = C, X_2 = D \text{ or } X_1 = D, X_2 = C) \\ &\quad + 2D(0)P(X_1 = D, X_2 = D). \end{aligned}$$

So we get

$$v(T) = \max\{2C(2), (C(1) + D(1)), 0\}.$$

Now we will check all possible cases.

(i) When $C(2) \leq 0$ and $C(1) + D(1) \leq 0$ hold, then we have $v(T) = 0$. Since $C(1) + D(1) \leq 0$ and I-1 imply $C(1) \leq 0$, we have $C(3) > 0$ by Assumption 2 (ii). Therefore we have $f(3)/3 = C(3) > v(T)/2$ which implies $v(N)/3 = \max\{f(m)/3\} \geq f(3)/3 > v(T)/2$ and Theorem 1 holds.

Next, we assume that

$$(*) \quad \max\{2C(2), (C(1) + D(1))\} > 0.$$

Case I-a. When $2C(2) - C(1) - D(1) = \beta_1 - 2\alpha_1 \geq 0$ holds, $v(T)$ can be expressed in the following form:

$$v(T) = 2C(2) = 2(\beta_1 - \alpha_1 - \alpha_0).$$

Since

$$f(3)/3 - v(T)/2 = C(3) - C(2) > 0 \text{ (by I-3)},$$

we have $v(N)/3 = \max\{f(m)\}/3 \geq f(3)/3 > v(T)/2$, which means that Theorem 1 holds.

Case I-b. When $2C(2) - C(1) - D(1) = \beta_1 - 2\alpha_1 \le 0$ holds, $v(T)$ can be expressed in the following form:

$$v(T) = C(1) + D(1) = \beta_1 - 2\alpha_0.$$

Since

$$\begin{aligned} f(1)/3 - v(T)/2 &= (C(1) + 2D(1))/3 - (C(1) + D(1))/2 \\ &= (D(1) - C(1))/6 = \beta_1/6 > 0 \ \text{(by Assumption 2(i))}, \end{aligned}$$

we have $v(N)/3 = \max\{f(m)\}/3 \ge f(1)/3 > v(T)/2$, which means that Theorem 1 holds. (End of the proof of Theorem 1).

Corollary 1. Under Assumptions 1 and 2, a no-conflict game ($\alpha_i < 0, ; i = 0, 1, 2$) is always strictly totally profitable. (It seems very natural consequence.)

Proof. In this case the games clearly belongs to Class I.

Corollary 2. Under Assumptions 1 and 2, all type of non-cooperative games classified by Muto([16]), i.e. Prisoner's dilemma games, Chicken games, Assurance games and Non-conflict games may possibly be strictly totally profitable in the cooperative game defined through the maximin strategy from the original non-cooperative games. Moreover, the cost for cooperation β_1, β_2 becomes bigger, that is, free ride may be more benefit, then the range of strictly total profitability becomes larger, which seems intuitively to be paradoxical.

Proof. It is obvious by cheking the condition of Class I.

4 Remarks

1. Under Assumptions 1 and 2, the following 3 cases never happen simultaneously, that is, it is not the case that $I_C \le I_D$ holds for any X_T under Assumptions 1 and 2,

ℓ-1: $\beta_1 - \alpha_0 \le 0$ ($\Longleftrightarrow D(1) \le 0$),

ℓ-2: $\beta_2 + \beta_1 - 2\alpha_1 \le 0$ ($\Longleftrightarrow C(2) + D(2) \le C(1) + D(1)$),

ℓ-3: $\beta_2 - \alpha_2 \le 0$ ($\Longleftrightarrow C(3) \le C(2)$).

Proof. By Assumptions 1, 2(i) and ℓ-1, $0 \ge D(1) > C(1)$ holds. By Assumption 2(i) and ℓ-1, 2, $0 \ge 2D(1) > C(1) + D(1) \ge C(2) + D(2) > 2C(2)$ holds. Also by ℓ-3 $0 \ge C(2) \ge C(3)$ holds. Therefore, we have $\max\{C(1), C(2), C(3)\} \le 0 = D(0)$ which contradics Assumption 2(ii).

2. If we assume Assumptions 1, 2(i) and ℓ-1,2,3, then the game is totally profitable.

Proof. In this case, for $T = \{1, 2\}$ we have

$$v(T) = \sup_{X_T} I_C.$$

Since

$$\begin{aligned} I_C &= 2C(3)P(X_1 = C,\, X_2 = C) \\ &\quad + (C(2) + D(2))P(X_1 = C,\, X_2 = D \text{ or } X_1 = D,\, X_2 = C) \\ &\quad + 2D(1)P(X_1 = D,\, X_2 = D) \end{aligned}$$

as is the estimation in the proof of Remark 1,

$$v(T) = \max\{2C(3),\, (C(2) + D(2)),\, 2D(1))\} \leq 0,$$

which follows $v(N)/3 \geq f(0)/3 = 0 \geq v(T)/2$.

The above case may happen in the situation that you are surrounded outside by enemies. In such case, it would be better off to stay all together than to run away left your friend(s).

3. Kimura([11]) discussed 4 cases of social dilemma situations in the context of Olson problems. The cooperative games defined from his all 4 cases through max-min principle are strictly totally profitable in our sense.

4. The relation between Simmel's three classifications and our formulation. When we see only the characteristic function of a cooperative game as Nakano did, if the game has non-empty core, then there would be no reason to segregate into a pair and an other, on the other hand, if the game has empty core, then there would be no reason to stay in one three person group. In contrast, in our formulation we investigate two different point of view simultaneously, group rationality under the maximin strategy and individual rationality assuming no cooperation each other. And in our point of view, what is Simmel's problem of three person group is a kind of conflicts or dilemma situations between these two rationality.

For examples, Simmel's *"the non-partisan and the mediator"* may correspond, in our formulation, to the situation that Ego is "C" but the other two are "D, D" and $\alpha_0 < 0,\ \alpha_1 > 0,\ \alpha_2 > 0$ (i.e. $m = 1$ is only Nash equilibrium and Chicken game in Muto's classification). Simmel's *"the tertius gaudens"* may correspond to the situation that Ego is "D" but the other two are "C, C" and $\alpha_0 < 0,\ \alpha_1 < 0,\ \alpha_2 > 0$ (i.e. $m = 2$ is only Nash equilibrium and Chicken game in Muto's classification). Simmel's *"divide et impera"* may correspond to the situation that all three are "D, D, D" and $\alpha_0 > 0,\ \alpha_1 > 0,\ \alpha_2 > 0$ (i.e. $m = 0$ is only Nash equilibrium and Prisoner's dilemma in Muto's classification). But in Simmel's classifications each member supposedly does not have equal power and ties asymmetrically but in our formulation they are in symmetric relations.

5. If Assumption 2 is not satisfied, say, $\beta_1 < 0$, $\beta_2 > 0$, there are a region of the parameters α_i ; $i = 0, 1, 2$ such that the game is not totally profitable(see the next section). These examples seem to show that without the Muto's cooperative situation([17]) (our Assumption 2) total profitability will be not guaranteed. On the contrary, under the Muto's cooperative situation, most case the games are totally profitable (Theorem 1). But the Muto's cooperative situation does not meen "all C strategy" $(m = n)$ is Pareto efficient nor Nash equilibrium. Therefore Theorem 1 will show many kinds of conflicts between group rationality and individual rationality.

6. We can not prove that whether or not only Assamption 1 and 2,(i),(ii) guarantee (strictly) total profitability. If it is true, Muto's conditions(our Assumption 2) might be very much meaningfull.

5 Examples of games which are not totally profitable

We assume that $D(0) = 0$ (Assumption 1) and the followings: (the case that for any X_T, $I_C \geq I_D$ holds)

I-1. $\beta_1 - \alpha_0 \geq 0$ $(\Longleftrightarrow D(1) \geq 0)$,

I-2. $\beta_2 + \beta_1 - 2\alpha_1 > 0$ $(\Longleftrightarrow C(2) + D(2) > C(1) + D(1))$,

I-3. $\beta_2 - \alpha_2 > 0$ $(\Longleftrightarrow C(3) > C(2))$,

I-b. $C(1) + D(1) > \max\{2C(2), 0\}$

Under the above conditions, we have

$$v(T) = C(1) + D(1) = \beta_1 - 2\alpha_0 > 0.$$

Moreover we asume

II-1. $\beta_1 < 0$,

II-2. $\alpha_2 > 0$,

II-3. $\beta_1 + \beta_2 - 2\alpha_1 = x\alpha_2$, $(x > 0)$,

II-4. $\beta_2 - \alpha_2 = y\alpha_2$, $(y > 0)$, where x, y are two auxiliary parameters.

Theorem 2. If a three person game (N, v) has the parameter (x, y) which satisfies $-x + 1 > y > 3x - 1$, then it is not totally profitable, (equivalently it has the empty core). Since by the conditions we have $\alpha_0 < 0$, $\alpha_2 > 0$, so the non-cooperative game is a chicken game. The range of (x, y) which satisfies the inequality is following Figure 1:

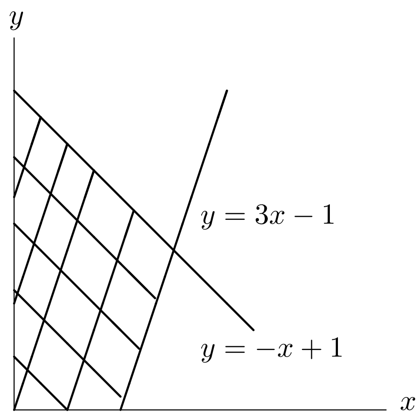

Figure 1.

Proof. First we remark that $v(T)/2 - f(0)/3 = v(T)/2 > 0$. By our Assumptions, we have

$$\begin{aligned} v(T)/2 - f(1)/3 &= (\beta_1 - 2\alpha_0)/2 - (2\beta_1 - 3\alpha_0)/3 \\ &= -\frac{\beta_1}{6} > 0. \end{aligned}$$

$$\begin{aligned} v(T)/2 - f(2)/3 &= (\beta_1 - 2\alpha_0)/2 - (\beta_2 + 3\beta_1 - 3\alpha_1 - 3\alpha_0)/3 \\ &= \frac{\beta_2}{6} - \frac{1}{2}(\beta_1 + \beta_2 - 2\alpha_1) \\ &= \frac{\alpha_2}{6}(y - 3x + 1). \end{aligned}$$

Therefore, we have
$v(T)/2 - f(2)/3 > 0 \Longleftrightarrow y > 3x - 1$.

$$\begin{aligned} v(T)/2 - f(3)/3 &= (\beta_1 - 2\alpha_0)/2 - (\beta_2 + \beta_1 - \alpha_2 - \alpha_1 - \alpha_0) \\ &= -\frac{1}{2}(\beta_2 - \alpha_2) + \frac{\alpha_2}{2} - \frac{x\alpha_2}{2} \\ &= \frac{\alpha_2}{2}(-y - x + 1), \end{aligned}$$

and
$v(T)/2 - f(3)/3 > 0 \Longleftrightarrow y < -x + 1$.

Therefore we conclude that if (x, y) satisfies $-x + 1 > y > 3x - 1$, then

$$\frac{v(T)}{2} > \max_{0 \leq m \leq 3} \frac{f(m)}{3} = \frac{v(N)}{3}$$

holds, which means that the game is not totally profitable.

6 Appendix. Balanced games([4])

Definition 4. A collection $\mathcal{B}$ of non-empty subsets of N is called a balanced collection if there exists positive number λ_T for all $T \in \mathcal{B}$ such that $\sum_{T\in\mathcal{B}} \lambda_T 1_T = 1$ holds, where 1_T is the indicator function of a set T, that is $1_T(t) = 1$ if $t \in T$ and $1_T(t) = 0$ if $t \notin T$.

Definition 5. A cooperative game (N, v) is called a balanced game if and only if for every balanced collection $\mathcal{B}$ with weights $\{\lambda_T\}_{T\in\mathcal{B}}$ the following holds:

$$\sum_{T\in\mathcal{B}} \lambda_T 1_T v(T) \leq v(N).$$

Propositon 2. (Theorem 1.3.5 of ([4])) A cooperative game has a non-empty core if and only if it is balanced.

Theorem 3. If a cooperative game (N, v) with $v(N) \geq 0$ is totally profitable, then it is balanced.

proof. Since $v(T) \leq |T|v(N)/|N|$, for every balanced collection $\mathcal{B}$ with weights $\{\lambda_T\}_{T\in\mathcal{B}}$, we have

$$\begin{aligned} \sum_{T\in\mathcal{B}} \lambda_T 1_T v(T) &\leq \sum_{T\in\mathcal{B}} \lambda_T 1_T v(N)|T|/|N| \\ &\leq \sum_{T\in\mathcal{B}} \lambda_T 1_T v(N) \quad \text{(since } v(N) \geq 0 \text{ and } |T|/|N| \leq 1) \\ &= v(N). \end{aligned}$$

In case of a symmetric game, this condition is also a necessary condition([29], p.27 Theorem 2.5).

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